

AN EQUIVALENCE BETWEEN DESINGULARIZED AND RENORMALIZED VALUES OF MULTIPLE ZETA FUNCTIONS AT NEGATIVE INTEGERS

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ABSTRACT. It is known that the special values of multiple zeta functions at non-positive arguments are indeterminate in most cases due to the occurrences of infinitely many singularities. In order to give a suitable rigorous meaning of the special values there, Furusho, Komori, Matsumoto and Tsumura introduced the desingularized values by the desingularization method to resolve all singularities. While, Ebrahimi-Fard, Manchon and Singer introduced the renormalized values to keep the “shuffle” relation by the renormalization procedure à la Connes and Kreimer. In this paper, we reveal an equivalence, that is, an explicit interrelationship between these two values. As a corollary, we also obtain an explicit formula to describe renormalized values in terms of Bernoulli numbers.

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0. INTRODUCTION

In 1776, Euler ([10]) considered a certain power series, the so-called double zeta values, and showed several relations among them. More than 200 years later Euler, in 1990s, the *multiple zeta value* (MZV for short) which is more general series

$$\zeta(k_1, \dots, k_n) := \sum_{0 < m_1 < \dots < m_n} \frac{1}{m_1^{k_1} \dots m_n^{k_n}}$$

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converging for $k_1, \dots, k_n \in \mathbb{N}$ and $k_n > 1$, came to be focused again by Hoffman ([14]) and Zagier ([20]). The MZV admits an iterated integral expression, which enables us to regard it as a period of a certain motive. ([7], [12] and [18]). MZV appears in calculations of the Kontsevich invariant in knot theory ([5] and [15]). MZV is also related to mathematical physics in [3] and [4]. They are explained in [22].

MZV is regarded as a special value at a positive integer point of the *multiple zeta-function* (MZF for short), the series

$$(0.1) \quad \zeta(s_1, \dots, s_n) := \sum_{0 < m_1 < \dots < m_n} \frac{1}{m_1^{s_1} \dots m_n^{s_n}}$$

which converges absolutely in the region

$$\{(s_1, \dots, s_n) \in \mathbb{C}^n \mid \Re(s_{n-k+1} + \dots + s_n) > k \ (1 \leq k \leq n)\}.$$

In the early 2000s, Zhao ([21]) and Akiyama, Egami and Tanigawa ([1]) independently showed that MZF can be meromorphically continued to \mathbb{C}^n . Especially, in [1], the set of all singularities of the function $\zeta(s_1, \dots, s_n)$ is determined as

$$(0.2) \quad \begin{aligned} s_n &= 1, \\ s_{n-1} + s_n &= 2, 1, 0, -2, -4, \dots, \\ s_{n-k+1} + \dots + s_n &= k - r \quad (3 \leq k \leq n, \ r \in \mathbb{N}_0). \end{aligned}$$

Because almost all of integer points with non-positive arguments are located in the above singularities, the special values of MZF there are indeterminate in all cases except for $\zeta(-k)$ at $k \in \mathbb{N}_0$, and $\zeta(-k_1, -k_2)$ at $k_1, k_2 \in \mathbb{N}_0$ with $k_1 + k_2$ odd. Actually, to give a nice definition of “ $\zeta(-k_1, \dots, -k_n)$ ” for $k_1, \dots, k_n \in \mathbb{N}_0$ is one of our most fundamental problems.

In order to resolve all infinitely many singularities of MZF, the desingularization method was introduced by Furusho, Komori, Matsumoto and Tsumura in [11]. By applying this method to $\zeta(s_1, \dots, s_n)$, they constructed the *desingularized MZF*¹ $\zeta_{\text{FKMT}}(s_1, \dots, s_n)$ which is entire on the whole space \mathbb{C}^n and they also showed its basic properties. The *desingularized value*

$$(0.3) \quad \zeta_{\text{FKMT}}(-k_1, \dots, -k_n) \in \mathbb{C}$$

is given as the special value of $\zeta_{\text{FKMT}}(s_1, \dots, s_n)$ at $(s_1, \dots, s_n) = (-k_1, \dots, -k_n)$ for $k_1, \dots, k_n \in \mathbb{N}_0$ (see Definition 1.4). In [11], its generating function given by

$$(0.4) \quad Z_{\text{FKMT}}(t_1, \dots, t_n) := \sum_{k_1, \dots, k_n=0}^{\infty} \frac{(-t_1)^{k_1} \dots (-t_n)^{k_n}}{k_1! \dots k_n!} \zeta_{\text{FKMT}}(-k_1, \dots, -k_n)$$

in $\mathbb{C}[[t_1, \dots, t_n]]$ was calculated and the desingularized values were described in terms of the Bernoulli numbers. (See Proposition 1.5.)

In contrast, Connes and Kreimer ([6]) started a Hopf algebraic approach to the renormalization procedure in the perturbative quantum field theory. A fundamental tool in their work is the *algebraic Birkhoff decomposition* (Theorem 2.6). By applying this decomposition to a certain Hopf algebra parameterizing regularized MZVs, Guo and Zhang ([13]) gave the *renormalized values* which satisfy the harmonic relations. Later, Manchon and Paycha ([17]) and Ebrahimi-Fard, Manchon and Singer

¹It is denoted by $\zeta_n^{\text{des}}((s_j); (1))$ in [11].

([9]) introduced the different renormalized values which obey harmonic(-like) relations by using different Hopf algebras. Meanwhile, Ebrahimi-Fard, Manchon and Singer ([8]) also introduced another type of the renormalized values (cf. Definition 2.8) satisfying the “shuffle relations” (see Proposition 2.10 for precise), which in this paper we denote as

$$(0.5) \quad \zeta_{\text{EMS}}(-k_1, \dots, -k_n) \in \mathbb{C}$$

for $k_1, \dots, k_n \in \mathbb{N}_0$, and which we consider with its generating function given by

$$(0.6) \quad Z_{\text{EMS}}(t_1, \dots, t_n) := \sum_{k_1, \dots, k_n=0}^{\infty} \frac{(-t_1)^{k_1} \dots (-t_n)^{k_n}}{k_1! \dots k_n!} \zeta_{\text{EMS}}(-k_1, \dots, -k_n)$$

in $\mathbb{C}[[t_1, \dots, t_n]]$.

Our main theorem in this paper is an equivalence between the desingularized values (0.3) and the renormalized values (0.5):

Theorem 3.5. *For $n \in \mathbb{N}$, we have*

$$Z_{\text{EMS}}(t_1, \dots, t_n) = \prod_{i=1}^n \frac{1 - e^{-t_i - \dots - t_n}}{t_i + \dots + t_n} \cdot Z_{\text{FKMT}}(-t_1, \dots, -t_n).$$

As a consequence of this theorem, the renormalized values can be given as linear combinations of the desingularized values and vice versa (cf. Example 3.7 and 3.8). By combining the above equivalence with the explicit formula (cf. Proposition 1.5) of the desingularized values shown in [11], we obtain the following explicit formula of the renormalized values.

Corollary 3.9. *For $k_1, \dots, k_n \in \mathbb{N}_0$, we have*

$$(0.7) \quad \zeta_{\text{EMS}}(-k_1, \dots, -k_n) = (-1)^{k_1 + \dots + k_n} \sum_{\substack{\nu_{1i} + \dots + \nu_{ii} = k_i \\ 1 \leq i \leq n}} \prod_{i=1}^n \frac{k_i!}{\prod_{j=i}^n \nu_{ij}!} \frac{B_{\nu_{ii} + \dots + \nu_{in} + 1}}{\nu_{ii} + \dots + \nu_{in} + 1}.$$

Here B_n is the Bernoulli number in (1.1).

The plan of our paper goes as follows. In section 1, we recall the desingularization method, desingularized MZF and the desingularized values introduced by Furusho, Komori, Matsumoto and Tsumura in [11]. In section 2, we review an algebraic framework on Hopf algebra in [8], and we prove an explicit formula of the reduced coproduct $\tilde{\Delta}_0$ (Proposition 2.5) which is required to prove the recurrence formula of renormalized values in [8] in section 3. We also review the algebraic Birkhoff decomposition and renormalized values in [8]. In section 3, by showing a recurrence formula (Proposition 3.3) we prove the above main results, that is, an equivalence between desingularized values and renormalized values (Theorem 3.5) and an explicit formula of renormalized values (Corollary 3.9).

1. DESINGULARIZATIONS

In this section, we review the desingularized values introduced by Furusho, Komori, Matsumoto and Tsumura in [11]. In §1.1 we recall the desingularization

method and desingularized MZF, and explain some remarkable properties of this function. In §1.2, we review the desingularized values and their generating function.

1.1. The desingularization method and desingularized MZFs. In this subsection, we review the desingularization method, the desingularized MZF. We also recall the basic properties of the desingularized MZF.

The desingularization method is a method to resolve all singularities of MZF. We recall the generating function² $\tilde{\mathfrak{H}}_n(t_1, \dots, t_n; c) \in \mathbb{C}[[t_1, \dots, t_n]]$ which is defined by in [11] Definition 1.9

$$\begin{aligned} \tilde{\mathfrak{H}}_n(t_1, \dots, t_n; c) &:= \prod_{j=1}^n \left(\frac{1}{\exp\left(\sum_{k=j}^n t_k\right) - 1} - \frac{c}{\exp\left(c \sum_{k=j}^n t_k\right) - 1} \right) \\ &= \prod_{j=1}^n \left(\sum_{m=1}^{\infty} (1 - c^m) B_m \frac{\left(\sum_{k=j}^n t_k\right)^{m-1}}{m!} \right) \end{aligned}$$

for $c \in \mathbb{R}$. Here B_m ($m \geq 0$) is the Bernoulli number which is defined by

$$(1.1) \quad \frac{x}{e^x - 1} := \sum_{m \geq 0} \frac{B_m}{m!} x^m.$$

We note that $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$.

Definition 1.1 ([11] Definition 3.1). For $s_1, \dots, s_n \in \mathbb{C}$, the *desingularized MZF* $\zeta_{\text{FKMT}}(s_1, \dots, s_n)$ is defined by

$$(1.2) \quad \zeta_{\text{FKMT}}(s_1, \dots, s_n) := \lim_{\substack{c \rightarrow 1 \\ c \in \mathbb{R} \setminus \{1\}}} \frac{1}{(1-c)^n} \prod_{k=1}^n \frac{1}{(e^{2\pi i s_k} - 1)\Gamma(s_k)} \int_{\mathcal{C}^n} \tilde{\mathfrak{H}}_n(t_1, \dots, t_n; c) \prod_{k=1}^n t_k^{s_k-1} dt_k.$$

Here \mathcal{C} is the path consisting of the positive real axis (top side), a circle around the origin of radius ε (sufficiently small), and the positive real axis (bottom side).

One of the remarkable properties of the desingularized MZF is that it is an entire function. According to the following proposition, the equation (1.2) is well-defined as the analytic function.

Proposition 1.2 ([11] Theorem 3.4). *The equation $\zeta_{\text{FKMT}}(s_1, \dots, s_n)$ can be analytically continued to \mathbb{C}^n as an entire function in $(s_1, \dots, s_n) \in \mathbb{C}^n$ by the following integral expression:*

$$\begin{aligned} &\zeta_{\text{FKMT}}(s_1, \dots, s_n) \\ &= \prod_{k=1}^n \frac{1}{(e^{2\pi i s_k} - 1)\Gamma(s_k)} \\ &\times \int_{\mathcal{C}^n} \prod_{j=1}^n \lim_{\substack{c \rightarrow 1 \\ c \in \mathbb{R} \setminus \{1\}}} \frac{1}{1-c} \left(\frac{1}{\exp\left(\sum_{k=j}^n t_k\right) - 1} - \frac{c}{\exp\left(c \sum_{k=j}^n t_k\right) - 1} \right) \prod_{k=1}^n t_k^{s_k-1} dt_k. \end{aligned}$$

²It is denoted by $\tilde{\mathfrak{H}}_n((t_j); (1); c)$ in [11].

We explain another remarkable properties of the desingularized MZF. For indeterminates u_j and v_j ($1 \leq j \leq n$), we set

$$\mathcal{G}((u_j), (v_j)) := \prod_{j=1}^n (1 - (u_j v_j + \cdots + u_n v_n)(v_j^{-1} - v_{j-1}^{-1}))$$

with the convention $v_0^{-1} := 0$, and we define the set of integers $\{a_{\mathbf{l}, \mathbf{m}}\}$ by

$$\mathcal{G}((u_j), (v_j)) = \sum_{\substack{\mathbf{l}=(l_j) \in \mathbb{N}_0^n \\ \mathbf{m}=(m_j) \in \mathbb{Z}^n \\ \sum_{j=1}^n m_j=0}} a_{\mathbf{l}, \mathbf{m}} \prod_{j=1}^n u_j^{l_j} v_j^{m_j}.$$

Another remarkable properties of the desingularized MZF is that the function is given by a finite ‘linear’ combination of MZFs.

Proposition 1.3 ([11] Theorem 3.8). *For $s_1, \dots, s_n \in \mathbb{C}$, we have*

$$\zeta_{\text{FKMT}}(s_1, \dots, s_n) = \sum_{\substack{\mathbf{l}=(l_j) \in \mathbb{N}_0^n \\ \mathbf{m}=(m_j) \in \mathbb{Z}^n \\ \sum_{j=1}^n m_j=0}} a_{\mathbf{l}, \mathbf{m}} \left(\prod_{j=1}^n (s_j)_{l_j} \right) \zeta(s_1 + m_1, \dots, s_n + m_n).$$

Here, $(s)_k$ is the Pochhammer symbols, that is, for $k \in \mathbb{N}$ and $s \in \mathbb{C}$ $(s)_0 := 1$ and $(s)_k := s(s+1) \cdots (s+k-1)$.

1.2. Desingularized values. We review the desingularized values and its explicit formula (Proposition 1.5), and then we give a recurrence formula of the desingularized values (Corollary 1.6).

The desingularized value is given as the special value at the integer points with non-positive arguments of an entire function:

Definition 1.4. For $k_1, \dots, k_n \in \mathbb{N}_0$, the *desingularized value* $\zeta_{\text{FKMT}}(-k_1, \dots, -k_n) \in \mathbb{C}$ is defined to be the special value of desingularized MZF $\zeta_{\text{FKMT}}(s_1, \dots, s_n)$ at $(s_1, \dots, s_n) = (-k_1, \dots, -k_n)$.

The generating function $Z_{\text{FKMT}}(t_1, \dots, t_n)$ of $\zeta_{\text{FKMT}}(-k_1, \dots, -k_n)$ in the equation (0.4) is explicitly calculated as follows.

Proposition 1.5 ([11] Theorem 3.7). *We have*

$$Z_{\text{FKMT}}(t_1, \dots, t_n) = \prod_{i=1}^n \frac{(1 - t_i - \cdots - t_n) e^{t_i + \cdots + t_n} - 1}{(e^{t_i + \cdots + t_n} - 1)^2}.$$

In terms of $\zeta_{\text{FKMT}}(-k_1, \dots, -k_n)$ for $-k_1, \dots, -k_n \in \mathbb{N}_0$, the above equation is reformulated to

(1.3)

$$\zeta_{\text{FKMT}}(-k_1, \dots, -k_n) = (-1)^{k_1 + \cdots + k_n} \sum_{\substack{\nu_{1i} + \cdots + \nu_{ni} = k_i \\ 1 \leq i \leq n}} \prod_{i=1}^n \frac{k_i!}{\prod_{j=i}^n \nu_{ij}!} B_{\nu_{ii} + \cdots + \nu_{in} + 1}.$$

By the above proposition we have the following recurrence formula:

Corollary 1.6.

$$(1.4) \quad Z_{\text{FKMT}}(t_1, \dots, t_n) = Z_{\text{FKMT}}(t_2, \dots, t_n) \cdot Z_{\text{FKMT}}(t_1 + \dots + t_n) \quad (n \in \mathbb{N}).$$

In terms of $\zeta_{\text{FKMT}}(-k_1, \dots, -k_n)$, the equation (1.4) is reformulated to

$$(1.5) \quad \zeta_{\text{FKMT}}(-k_1, \dots, -k_n) = \sum_{\substack{i_2 + j_2 = k_2 \\ \vdots \\ i_n + j_n = k_n}} \prod_{a=2}^n \binom{k_a}{i_a} \zeta_{\text{FKMT}}(-i_2, \dots, -i_n) \zeta_{\text{FKMT}}(-k_1 - j_2 - \dots - j_n)$$

for $k_1, \dots, k_n \in \mathbb{N}_0$. Here we use $\binom{k_a}{i_a} := \frac{k_a!}{i_a!(k_a - i_a)!}$.

In §3, we will show that the same formula as (1.5) holds for the renormalized value $\zeta_{\text{EMS}}(-k_1, \dots, -k_n)$ in the equation (3.8).

2. REMORMALIZATIONS

In this section, we recall the renormalization procedure to define renormalized values which is introduced by Ebrahimi-Fard, Manchon and Singer. In §2.1, we start by recalling their framework of a Hopf algebra generated by words and in §2.2 we show an explicit formula in Proposition 2.5 to calculate the reduced coproduct $\tilde{\Delta}_0$. This proposition is essential to show the recurrence formula of $\zeta_{\text{EMS}}(-k_1, \dots, -k_n)$ in §3. In §2.3 we explain the algebraic Birkhoff decomposition à la Connes and Kreimer which is required to define renormalized values.

2.1. Algebraic frameworks. We follow the conventions of [8]. Let $X_0 := \{j, d, y\}$ be the set of three elements j , d and y . Let W_0 be the associative monoid, with the empty word $\mathbf{1}$ as a unit, generated by X_0 with the rule $jd = dj = \mathbf{1}$. Any element $w \in W_0$ can be uniquely represented by

$$w = j^{k_1} y \cdots j^{k_n} y$$

for $k_1, \dots, k_n \in \mathbb{Z}$. An element of W_0 is called a *word*. Put $Y_0 := W_0 y \cup \{\mathbf{1}\}$ and we call an element of Y_0 *admissible*. We denote the \mathbb{Q} -linear space \mathcal{A}_0 generated by W_0 by $\mathcal{A}_0 := \langle W_0 \rangle_{\mathbb{Q}}$. The linear space \mathcal{A}_0 is naturally equipped with a structure of a non-commutative algebra. We equip this \mathcal{A}_0 with a new product $\sqcup_0 : \mathcal{A}_0 \otimes \mathcal{A}_0 \rightarrow \mathcal{A}_0$ which is a \mathbb{Q} -linear map recursively defined by

$$\begin{aligned} \mathbf{1} \sqcup_0 w &:= w \sqcup_0 \mathbf{1} := w \quad (w \in W_0), \\ yu \sqcup_0 v &:= u \sqcup_0 yv := y(u \sqcup_0 v) \quad (u, v \in W_0), \\ ju \sqcup_0 jv &:= j(u \sqcup_0 jv) + j(ju \sqcup_0 v) \quad (u, v \in W_0), \\ du \sqcup_0 dv &:= d(u \sqcup_0 dv) - u \sqcup_0 d^2 v \quad (u, v \in W_0). \end{aligned}$$

Then $(\mathcal{A}_0, \sqcup_0)$ forms a unitary, nonassociative, noncommutative \mathbb{Q} -algebra. We define

$$\mathcal{T} := \langle \{j^{k_1} y \cdots j^{k_{n-1}} y j^{k_n} \in W_0 \mid k_n \neq 0, n \in \mathbb{N}\} \rangle_{\mathbb{Q}},$$

that is, to be the linear subspace of \mathcal{A}_0 linearly generated by words ending in d or j and

$$\mathcal{L} := \langle j^k \{d(u \sqcup_0 v) - du \sqcup_0 v - u \sqcup_0 dv\} \mid k \in \mathbb{Z}, u, v \in W_0 y \rangle_{(\mathcal{A}_0, \sqcup_0)},$$

that is, to be the two-sided ideal of $(\mathcal{A}_0, \sqcup_0)$ algebraically generated by the above elements. The subspace \mathcal{T} forms a two-sided ideal of \mathcal{A}_0 by [8] Lemma 3.4. We define the quotient algebra

$$\mathcal{B}'_0 := \mathcal{A}_0 / (\mathcal{T} + \mathcal{L}).$$

We consider the map

$$(2.1) \quad \zeta_t^{\sqcup} : \mathcal{B}'_0 \rightarrow \mathbb{Q}[[t]]$$

by $\zeta_t^{\sqcup}(\mathbf{1}) := 1$ and for $k_1, \dots, k_n \in \mathbb{Z}$,

$$\zeta_t^{\sqcup}(j^{k_n} y \cdots j^{k_1} y) := \text{Li}_{k_1, \dots, k_n}(t).$$

Here $\text{Li}_{k_1, \dots, k_n}(t)$ is the *multiple polylogarithm* defined by

$$\text{Li}_{k_1, \dots, k_n}(t) := \sum_{0 < m_1 < \dots < m_n} \frac{t^{m_n}}{m_1^{k_1} \cdots m_n^{k_n}}.$$

Lemma 2.1. *The map ζ_t^{\sqcup} is well-defined and forms an algebra homomorphism.*

The first half of the claim of Lemma 2.1 is proved in the same way to proof of [8] Proposition 3.5 and the latter half of the claim of Lemma 2.1 is proved in [8] Lemma 3.6.

Remark 2.2. The restriction of the shuffle product \sqcup_0 to admissible words at positive arguments corresponds the usual shuffle product \sqcup as is proved in [8] Lemma 3.7. Let $\mathcal{C} := \mathbb{Q} \oplus j\mathbb{Q}\langle j, y \rangle y$ and $\mathcal{D} := \mathbb{Q} \oplus x_0\mathbb{Q}\langle x_0, x_1 \rangle x_1$. Then two algebras (\mathcal{C}, \sqcup_0) and (\mathcal{D}, \sqcup) become isomorphic under the linear map $\Phi : (\mathcal{D}, \sqcup) \rightarrow (\mathcal{C}, \sqcup_0)$ by $\Phi(\mathbf{1}) := \mathbf{1}$ and for $k_1, \dots, k_n \in \mathbb{N}$ with $k_1 > 1$,

$$\Phi(x_0^{k_1-1} x_1 \cdots x_0^{k_n-1} x_1) := j^{k_1-1} y \cdots j^{k_n-1} y.$$

Let $L := \{d, y\}$ be the set of two elements d and y . Let L^* be the free monoid of L with empty word $\mathbf{1}$ as a unit. This L^* forms a submonoid of W_0 . Put $Y := L^* y \cup \{\mathbf{1}\} \subset Y_0$. So all elements of Y are admissible. The *weight* $\text{wt}(w)$ of a word $w \in L^*$ means the number of letters appearing in w and the *depth* $\text{dp}(w)$ of a word $w \in L^*$ is given by the number of y appearing in w . We denote the free unitary, associative, noncommutative \mathbb{Q} -algebra of L by $\mathbb{Q}\langle L \rangle$. Then $(\mathbb{Q}\langle L \rangle, \sqcup_0)$ forms a unitary, nonassociative, noncommutative \mathbb{Q} -subalgebra of \mathcal{A}_0 . The algebra $\mathbb{Q}\langle L \rangle$ also forms a counital, cocommutative coalgebra. (See [8] §3.3.5.) We define

$$\mathcal{T}_- := \langle \{wd \mid w \in L^*\} \rangle_{\mathbb{Q}} (= \mathcal{T} \cap \mathbb{Q}\langle L \rangle),$$

that is, to be the linear subspace of $\mathbb{Q}\langle L \rangle$ linearly generated by words ending in d and

$$\mathcal{L}_- := \langle d^k \{d(u \sqcup_0 v) - du \sqcup_0 v - u \sqcup_0 dv\} \mid k \in \mathbb{N}_0, u, v \in L^* \rangle_{(\mathbb{Q}\langle L \rangle, \sqcup_0)},$$

that is, to be the two-sided ideal of $(\mathbb{Q}\langle L \rangle, \sqcup_0)$ algebraically generated by the above elements. We consider the \mathbb{Q} -linear subspace

$$\mathcal{S}_- := \mathcal{T}_- + \mathcal{L}_-$$

of $\mathbb{Q}\langle L \rangle$ generated by \mathcal{L}_- and \mathcal{T}_- . This \mathcal{S}_- also forms a two-sided ideal as our previous $\mathcal{T} + \mathcal{L}$. We put the quotient

$$\mathcal{H}_0 := \mathbb{Q}\langle L \rangle / \mathcal{S}_-.$$

Actually \mathcal{H}_0 forms a connected, filtered, commutative and cocommutative Hopf algebra (cf. [8] §3.3.6), whose product is equal to \sqcup_0 and whose coproduct is given by

$$\Delta_0(w) := \sum_{\substack{S \subset [n] \\ S: \text{admissible}}} w_S \otimes w_{\overline{S}},$$

for $w \in Y \setminus \{\mathbf{1}\} \subset \mathcal{H}_0$. In the summation, S may be empty. we put $n := \text{wt}(w)$, $[n] := \{1, \dots, n\}$ and $\overline{S} := [n] \setminus S$. For $w := x_1 \cdots x_n$ ($x_i \in L^*$, $i = 1, \dots, n$) and $S := \{i_1, \dots, i_k\}$ with $1 \leq i_1 < \dots < i_k \leq n$, we define $w_S := x_{i_1} \cdots x_{i_k}$. We call the set S *admissible* if both $w_S, w_{\overline{S}} \in Y$. See [8] §3.3.8 for combinatorial method using polygon to compute $\Delta_0(w)$. We define \mathbb{Q} -linear map $\tilde{\Delta}_0 : \mathcal{H}_0 \rightarrow \mathcal{H}_0 \otimes \mathcal{H}_0$ by

$$(2.2) \quad \tilde{\Delta}_0(w) := \Delta_0(w) - 1 \otimes w - w \otimes 1 \quad (w \in Y),$$

and we call $\tilde{\Delta}_0$ the *reduced product*.

2.2. An explicit formula of the reduced coproduct $\tilde{\Delta}_0$. We show an explicit formula (Proposition 2.5) to calculate the reduced coproduct $\tilde{\Delta}_0$ in this subsection. This proposition is important to prove the recurrence formula of $\zeta_{\text{EMS}}(-k_1, \dots, -k_n)$ in §3.

We consider the bilinear map $f : \mathbb{Q}\langle L \rangle \times \mathbb{Q}\langle L \rangle^{\otimes 2} \rightarrow \mathbb{Q}\langle L \rangle^{\otimes 2}$ defined by

$$\begin{aligned} f(\mathbf{1}, w \otimes w') &:= w \otimes w', \\ f(d, w \otimes w') &:= dw \otimes w' + w \otimes dw', \\ f(y, w \otimes w') &:= yw \otimes w' + w \otimes yw', \end{aligned}$$

and inductively

$$f(xx_0, w \otimes w') := f(x, f(x_0, w \otimes w')),$$

for $w, w' \in \mathbb{Q}\langle L \rangle$, $x_0 \in L$ and $x \in L^*$. Then the following lemma holds:

Lemma 2.3. *There is a map $\overline{f} : \mathbb{Q}\langle L \rangle \times \mathcal{H}_0^{\otimes 2} \rightarrow \mathcal{H}_0^{\otimes 2}$ which makes the following diagram commutative:*

$$\begin{array}{ccc} \mathbb{Q}\langle L \rangle \otimes \mathbb{Q}\langle L \rangle & \xrightarrow{f(x, \cdot)} & \mathbb{Q}\langle L \rangle \otimes \mathbb{Q}\langle L \rangle \\ \pi \downarrow & & \downarrow \pi \\ \mathcal{H}_0 \otimes \mathcal{H}_0 & \xrightarrow{\overline{f}(x, \cdot)} & \mathcal{H}_0 \otimes \mathcal{H}_0 \end{array}$$

where $x \in \mathbb{Q}\langle L \rangle$ and $\pi : \mathbb{Q}\langle L \rangle^{\otimes 2} \rightarrow \mathcal{H}_0^{\otimes 2}$ is a natural projection.

Proof. It is sufficient to prove $f(x, \ker \pi) \subset \ker \pi$ for $x \in L^*$. Here $\ker \pi = \mathbb{Q}\langle L \rangle \otimes \mathcal{S}_- + \mathcal{S}_- \otimes \mathbb{Q}\langle L \rangle$. We show this by induction on $\text{wt}(x)$. Let $x_0 = d$ or y and put $v \in \mathcal{S}_-$. If $v \in \mathcal{T}_-$, it is clear that $x_0 v \in \mathcal{T}_- \subset \mathcal{S}_-$. If $v \in \mathcal{L}_-$, for $x_0 = d$ it is easy to see that $dv \in \mathcal{L}_- \subset \mathcal{S}_-$ by the definition of \mathcal{L}_- . Because \mathcal{L}_- is a two-sided ideal of $(\mathbb{Q}\langle L \rangle, \sqcup_0)$, we have $y \sqcup_0 v \in \mathcal{L}_-$ for $x_0 = y$. By the definition of \sqcup_0 , we get

$$y \sqcup_0 v = y(1 \sqcup_0 v) = yv \in \mathcal{L}_- \subset \mathcal{S}_-.$$

Because \mathcal{S}_- is $\mathcal{L}_- + \mathcal{T}_-$, for $v \in \mathcal{S}_-$ and $x_0 = d$ or y , we have $x_0 v \in \mathcal{S}_-$.

Let $w \in L^*$ and $v \in \mathcal{S}_-$. Then $x_0 v \in \mathcal{S}_-$, so we have

$$\begin{aligned}\pi(f(x_0, w \otimes v)) &= \pi(x_0 w \otimes v + w \otimes x_0 v) \\ &= \pi(x_0 w \otimes v) + \pi(w \otimes x_0 v) \\ &= 0.\end{aligned}$$

Let $w \in L^*$ and $v \in \mathcal{S}_-$. For $x \in L^*$, we get

$$\begin{aligned}\pi(f(x x_0, w \otimes v)) &= \pi(f(x, f(x_0, w \otimes v))) \\ &= \pi(f(x, x_0 w \otimes v + w \otimes x_0 v)) \\ &= \pi(f(x, x_0 w \otimes v)) + \pi(f(x, w \otimes x_0 v)).\end{aligned}$$

By our induction assumption,

$$= 0.$$

This also applies to the case when $w \in \mathcal{S}_-$ and $v \in L^*$, so the claim holds. \square

For $x \in L^*$ and $w, w' \in Y$, we simply denote $\overline{f}(x, w \otimes w')$ by $x \bullet (w \otimes w')$ and we define

$$w \otimes_{\text{sym}} w' := w \otimes w' + w' \otimes w \in \mathcal{H}_0 \otimes \mathcal{H}_0.$$

Then, the following equations hold in $\mathcal{H}_0 \otimes \mathcal{H}_0$:

$$(2.3) \quad d^n \bullet (w \otimes_{\text{sym}} w') = \sum_{i+j=n} \binom{n}{i} d^i w \otimes_{\text{sym}} d^j w',$$

$$(2.4) \quad (d^n y) \bullet (w \otimes_{\text{sym}} w') = \sum_{i+j=n} \binom{n}{i} \sum_{\{u,v\}=\{d^i y, d^j\}} u w \otimes_{\text{sym}} v w',$$

for $n \in \mathbb{N}$, $w, w' \in Y$. These equations can be proved inductively on $n \in \mathbb{N}$.

Proposition 2.4. *For $w \in Y \setminus \{1\}$,*

$$(2.5) \quad \tilde{\Delta}_0(dw) = d \bullet \tilde{\Delta}_0(w),$$

$$(2.6) \quad \tilde{\Delta}_0(yw) = y \bullet \tilde{\Delta}_0(w) + y \otimes_{\text{sym}} w.$$

Proof. Let w be in $Y \setminus \{1\}$. By the definition of Δ_0 and the equation (2.2), we have

$$\begin{aligned}
\tilde{\Delta}_0(dw) &= \Delta_0(dw) - 1 \otimes_{\text{sym}} dw \\
&= \sum_{\substack{S \subset [n+1] \\ S: \text{admissible}}} (dw)_S \otimes (dw)_{\overline{S}} - 1 \otimes_{\text{sym}} dw \\
&= \sum_{\substack{1 \in S \subset [n+1] \\ S: \text{admissible}}} (dw)_S \otimes (dw)_{\overline{S}} + \sum_{\substack{1 \notin S \subset [n+1] \\ S: \text{admissible}}} (dw)_S \otimes (dw)_{\overline{S}} - 1 \otimes_{\text{sym}} dw \\
&= \sum_{\substack{S \subset [n] \\ S: \text{admissible}}} d \cdot w_S \otimes w_{\overline{S}} + \sum_{\substack{S \subset [n] \\ S: \text{admissible}}} w_S \otimes d \cdot w_{\overline{S}} - (d \otimes_{\text{sym}} w + 1 \otimes_{\text{sym}} dw) \\
&= \sum_{\substack{S \subset [n] \\ S: \text{admissible}}} (d \cdot w_S \otimes w_{\overline{S}} + w_S \otimes d \cdot w_{\overline{S}}) - (d \otimes_{\text{sym}} w + 1 \otimes_{\text{sym}} dw) \\
&= d \bullet \left(\sum_{\substack{S \subset [n] \\ S: \text{admissible}}} w_S \otimes w_{\overline{S}} - 1 \otimes_{\text{sym}} w \right) \\
&= d \bullet \tilde{\Delta}_0(w).
\end{aligned}$$

We use $d \otimes_{\text{sym}} w = 0$ in $\mathcal{H}_0 \otimes \mathcal{H}_0$ at the fourth equality. The equation (2.6) can be proved in the same way. \square

Proposition 2.5. *Let $w_m := d^m y$ for $m \in \mathbb{N}_0$. Then for $n \in \mathbb{N}_{\geq 2}$ and $k_1, \dots, k_n \in \mathbb{N}_0$, we have*

(2.7)

$$\begin{aligned}
\tilde{\Delta}_0(w_{k_1} \cdots w_{k_n}) &= \sum_{i_1 + j_1 = k_1} \binom{k_1}{i_1} d^{i_1} y \otimes_{\text{sym}} d^{j_1} w_{k_2} \cdots w_{k_n} \\
&+ \sum_{p=2}^{n-1} \sum_{\substack{i_1 + j_1 = k_1 \\ \vdots \\ i_p + j_p = k_p}} \prod_{a=1}^p \binom{k_a}{i_a} \sum_{\substack{\{u_q, v_q\} = \{d^{i_q}, d^{j_q} y\} \\ 1 \leq q \leq p-1}} (u_1 \cdots u_{p-1} d^{i_p} y \otimes_{\text{sym}} v_1 \cdots v_{p-1} d^{j_p} w_{k_{p+1}} \cdots w_{k_n}).
\end{aligned}$$

Here $\{u_q, v_q\} = \{d^{i_q}, d^{j_q} y\}$ means $(u_q, v_q) = (d^{i_q}, d^{j_q} y)$ or $(d^{j_q} y, d^{i_q})$.

Proof. Because we have

$$(2.8) \quad \tilde{\Delta}_0(d^a y w) = d^a \bullet \left(y \otimes_{\text{sym}} w + y \bullet \tilde{\Delta}_0(w) \right) \quad (a \in \mathbb{N}_0)$$

by Proposition 2.4, we compute

$$\begin{aligned}
&\tilde{\Delta}_0(w_{k_1} w_{k_2} \cdots w_{k_n}) \\
&= d^{k_1} \bullet (y \otimes_{\text{sym}} w_{k_2} \cdots w_{k_n}) + (d^{k_1} y) \bullet \tilde{\Delta}_0(w_{k_2} \cdots w_{k_n}) \\
&= d^{k_1} \bullet (y \otimes_{\text{sym}} w_{k_2} \cdots w_{k_n}) + (d^{k_1} y d^{k_2}) \bullet (y \otimes_{\text{sym}} w_{k_3} \cdots w_{k_n}) \\
&\quad + (d^{k_1} y d^{k_2} y) \bullet \tilde{\Delta}_0(w_{k_3} \cdots w_{k_n}).
\end{aligned}$$

By using the equation (2.8) repeatedly, we get

$$\begin{aligned} &= \sum_{p=1}^{n-1} (d^{k_1} y \cdots y d^{k_p}) \bullet (y \otimes_{\text{sym}} w_{k_{p+1}} \cdots w_{k_n}) \\ &\quad + (d^{k_1} y \cdots d^{k_{n-1}} y) \bullet \tilde{\Delta}_0(w_{k_n}). \end{aligned}$$

Because $\tilde{\Delta}_0(d^a y) = 0$ ($a \in \mathbb{N}_0$) by the definition of $\tilde{\Delta}_0$, the second term vanishes. Therefore by (2.3), we get

$$\begin{aligned} &\tilde{\Delta}_0(w_{k_1} w_{k_2} \cdots w_{k_n}) \\ &= \sum_{p=1}^{n-1} (d^{k_1} y \cdots d^{k_{p-1}} y) \bullet \left(\sum_{i_p + j_p = k_p} \binom{k_p}{i_p} d^{i_p} y \otimes_{\text{sym}} d^{j_p} w_{k_{p+1}} \cdots w_{k_n} \right). \end{aligned}$$

And by using (2.4) repeatedly, we have

$$\begin{aligned} &= \sum_{i_1 + j_1 = k_1} \binom{k_1}{i_1} d^{i_1} y \otimes_{\text{sym}} d^{j_1} w_{k_2} \cdots w_{k_n} \\ &\quad + \sum_{p=2}^{n-1} \sum_{\substack{i_1 + j_1 = k_1 \\ \vdots \\ i_p + j_p = k_p}} \prod_{a=1}^p \binom{k_a}{i_a} \sum_{\substack{\{u_q, v_q\} = \{d^{i_q}, d^{j_q} y\} \\ 1 \leq q \leq p-1}} (u_1 \cdots u_{p-1} d^{i_p} y \otimes_{\text{sym}} v_1 \cdots v_{p-1} d^{j_p} w_{k_{p+1}} \cdots w_{k_n}). \end{aligned}$$

□

2.3. The algebraic Birkhoff decomposition and renormalized values. We explain the algebraic Birkhoff decomposition. This decomposition is a fundamental tool in a work of Connes and Kreimer [6] on their Hopf algebraic approach to renormalization of perturbative quantum field theory. This decomposition is necessary to define renormalized values.

Based on [16], we recall the algebraic Birkhoff decomposition. We denote the product and the unit of \mathbb{Q} -algebra \mathcal{A} by $m_{\mathcal{A}}$ and $u_{\mathcal{A}}$. For a Hopf algebra \mathcal{H} over \mathbb{Q} , we mean $\Delta_{\mathcal{H}}$, $\varepsilon_{\mathcal{H}}$ and $S_{\mathcal{H}}$ to be its coproduct, its counit and its antipode respectively. In this paper, we often use Sweedler's notation:

$$(2.9) \quad \tilde{\Delta}_0(w) := \sum_{(w)} w' \otimes w''.$$

Let \mathcal{H} be a Hopf algebra over \mathbb{Q} , \mathcal{A} be a \mathbb{Q} -algebra and $\mathcal{L}(\mathcal{H}, \mathcal{A})$ be the set of \mathbb{Q} -linear maps from \mathcal{H} to \mathcal{A} . We define the *convolution* $\phi * \psi \in \mathcal{L}(\mathcal{H}, \mathcal{A})$ by

$$\phi * \psi := m_{\mathcal{A}} \circ (\phi \otimes \psi) \circ \Delta_{\mathcal{H}}$$

for \mathbb{Q} -linear maps ϕ and $\psi \in \mathcal{L}(\mathcal{H}, \mathcal{A})$. Let \mathcal{H} be a Hopf algebra over \mathbb{Q} and \mathcal{A} be a \mathbb{Q} -algebra. The subset

$$G(\mathcal{H}, \mathcal{A}) := \{\phi \in \mathcal{L}(\mathcal{H}, \mathcal{A}) \mid \phi(1) = 1_{\mathcal{A}}\}$$

endowed with the above convolution product $*$ forms a group. The unit is given by a map $e = u_{\mathcal{A}} \circ \varepsilon_{\mathcal{H}}$ and the inverse of $\phi \in G(\mathcal{H}, \mathcal{A})$ is given by $\phi^{-1} = \phi S_{\mathcal{H}}$.

Let \mathcal{H} be a connected filtered Hopf algebra over \mathbb{Q} , that is, \mathcal{H} has a filtration of \mathbb{Q} -linear subspace:

$$\mathcal{H}^0 \subset \mathcal{H}^1 \subset \cdots \subset \mathcal{H}^n \subset \bigcup_{n \in \mathbb{N}_0} \mathcal{H}^n = \mathcal{H}$$

with $\mathcal{H}^0 = \mathbb{Q}$ and with the conditions: $\mathcal{H}^m \mathcal{H}^n \subset \mathcal{H}^{m+n}$ and $S_{\mathcal{H}}(\mathcal{H}^n) \subset \mathcal{H}^n$ and $\Delta_{\mathcal{H}}(\mathcal{H}^n) \subset \sum_{p+q=n} \mathcal{H}^p \otimes \mathcal{H}^q$ for $m, n \in \mathbb{N}_0$.

Let $\mathcal{A} := \mathbb{Q}[\frac{1}{z}, z] := \mathbb{Q}[[z]][\frac{1}{z}]$ be the algebra consisting of all Laurent series. And we decompose it as $\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+$ where $\mathcal{A}_- := \frac{1}{z}\mathbb{Q}[\frac{1}{z}]$ and $\mathcal{A}_+ := \mathbb{Q}[[z]]$. We define a projection $\pi : \mathcal{A} \rightarrow \mathcal{A}_-$ by

$$\pi \left(\sum_{n=-k}^{\infty} a_n z^n \right) := \sum_{n=-k}^{-1} a_n z^n,$$

with $a_n \in \mathbb{Q}$ and $k \in \mathbb{Z}$. Here we use the convention the sum over empty set is zero.

The following theorem is the fundamental tool of Connes and Kreimer ([6]) in the renormalization procedure of perturbative quantum field theory.

Theorem 2.6 ([6], [8], [16]: **algebraic Birkhoff decomposition**). *For $\phi \in G(\mathcal{H}, \mathcal{A})$, there are unique linear maps $\phi_+ : \mathcal{H} \rightarrow \mathcal{A}_+$ and $\phi_- : \mathcal{H} \rightarrow \mathbb{Q} \oplus \mathcal{A}_-$ with $\phi_-(1) = 1 \in \mathbb{Q}$ such that*

$$\phi = \phi_-^{-1} * \phi_+.$$

Moreover the maps ϕ_- and ϕ_+ are algebra homomorphisms if ϕ is an algebra homomorphism.

We define the \mathbb{Q} -linear map $\phi : \mathcal{H}_0 \rightarrow \mathcal{A}$ by $\phi(1) := 1$ and for $k_1, \dots, k_n \in \mathbb{N}_0$,

$$(2.10) \quad d^{k_1} y \cdots d^{k_n} y \mapsto \phi(d^{k_1} y \cdots d^{k_n} y)(z) := \partial_z^{k_1} (x \partial_z^{k_2}) \cdots (x \partial_z^{k_n}) (x(z))$$

where $x := x(z) := \frac{e^z}{1-e^z} \in \mathcal{A}$ and ∂_z is the derivative by z .

Proposition 2.7 ([8] §4.2). *The \mathbb{Q} -linear map $\phi : \mathcal{H}_0 \rightarrow \mathcal{A}$ is well-defined and forms algebra homomorphism. Moreover, the following diagram is commutative:*

$$\begin{array}{ccc} (\mathcal{H}_0, \sqcup_0) & \xrightarrow{\zeta_t^{\sqcup}} & (\mathbb{Q}[[t]], \cdot) \\ & \searrow \phi & \downarrow t \mapsto e^z \\ & & (\mathcal{A}, \cdot) \end{array}$$

where ζ_t^{\sqcup} is the map in (2.1).

Because the map ϕ is algebraic by the above proposition, we obtain the algebraic map:

$$(2.11) \quad \phi_+ : \mathcal{H}_0 \rightarrow \mathcal{A}_+$$

which is an algebra homomorphism by Theorem 2.6.

Definition 2.8 ([8] §4.2). The *renormalized value* ³ $\zeta_{\text{EMS}}(-k_1, \dots, -k_n)$ is defined by

$$(2.12) \quad \zeta_{\text{EMS}}(-k_1, \dots, -k_n) := \lim_{z \rightarrow 0} \phi_+(d^{k_n} y \cdots d^{k_1} y)(z)$$

³If we follow the notations of [8], it should be denoted by $\zeta_+(-k_n, \dots, k_1)$.

for $k_1, \dots, k_n \in \mathbb{N}_0$.

One of the remarkable properties of the renormalized values is to be equal to special values of the meromorphic continuation of MZFs at non-positive arguments which do not locate at their singularities.

Proposition 2.9 ([8] Theorem 4.3). *For $k_1 \in \mathbb{N}_0$, we have*

$$\zeta_{\text{EMS}}(-k_1) = \zeta(-k_1)$$

and for $k_1, k_2 \in \mathbb{N}_0$ with $k_1 + k_2$ odd, we have

$$\zeta_{\text{EMS}}(-k_1, -k_2) = \zeta(-k_1, -k_2).$$

We remind that, as is showed in the set (0.2), $\zeta(s_1, \dots, s_n)$ is always irregular at $(s_1, \dots, s_n) = (-k_1, \dots, -k_n) \in \mathbb{Z}_{\leq 0}^n$ for $n \geq 3$.

Another remarkable property of the renormalized values is that a certain shuffle relation hold for them. Because \sqcup_0 is the product of \mathcal{H}_0 and $\phi_+ : \mathcal{H}_0 \rightarrow \mathbb{Q}[[z]]$ is algebraic by Theorem 2.6, we obtain the following proposition:

Proposition 2.10 ([8] §4.2: **shuffle relation**). *For $w, w' \in Y$, we have*

$$\phi_+(w \sqcup_0 w') = \phi_+(w)\phi_+(w').$$

Here are examples in lower depth:

Examples 2.11. For $a, b, c \in \mathbb{N}_0$, we have

$$\begin{aligned} \zeta_{\text{EMS}}(-a) \cdot \zeta_{\text{EMS}}(-b) &= \begin{cases} \sum_{k=0}^a (-1)^k \binom{a}{k} \zeta_{\text{EMS}}(-b-k, -a+k) & \text{if } b \geq 1, \\ \zeta_{\text{EMS}}(-a, 0) & \text{if } b = 0, \end{cases} \\ \zeta_{\text{EMS}}(-a) \cdot \zeta_{\text{EMS}}(-b, -c) &= \begin{cases} \sum_{k=0}^c (-1)^k \binom{c}{k} \zeta_{\text{EMS}}(-b, -c-k, -a+k) & \text{if } c \geq 1, \\ \sum_{k=0}^c (-1)^k \binom{c}{k} \zeta_{\text{EMS}}(-b-k, -a+k, 0) & \text{if } b \geq 1, c = 0, \\ \zeta_{\text{EMS}}(-a, 0, 0) & \text{if } b = c = 0. \end{cases} \end{aligned}$$

For our comparison, we remind below the usual shuffle relation for positive arguments. For $a, b \in \mathbb{N}_{>1}$,

$$\zeta(a) \cdot \zeta(b) = \sum_{k=0}^{a-1} \binom{b-1+k}{k} \zeta(a-k, b+k) + \sum_{k=0}^{b-1} \binom{a-1+k}{k} \zeta(b-k, a+k),$$

and for $a, c \in \mathbb{N}_{>1}$ and $b \in \mathbb{N}$,

$$\begin{aligned} \zeta(a) \cdot \zeta(b, c) &= \sum_{k=0}^{a-1} \sum_{i=0}^{a-k-1} \binom{c-1+k}{k} \binom{b-1+i}{i} \zeta(a-k-i, b+i, c+k) \\ &\quad + \sum_{k=0}^{a-1} \sum_{j=0}^{b-1} \binom{c-1+k}{k} \binom{a-k-1+j}{j} \zeta(b-j, a-k+j, c+k) \\ &\quad + \sum_{k=0}^{c-1} \binom{a-1+k}{k} \zeta(b, c-k, a+k). \end{aligned}$$

3. MAIN RESULTS

In this section, we prove a recurrence formula among renormalized values of MZFs in Proposition 3.3. Moreover, by showing that the renormalized value $\zeta_{\text{EMS}}(-k_1, \dots, -k_n)$ satisfies the recurrence formula similar to the one (1.5) for $\zeta_{\text{FKMT}}(-k_1, \dots, -k_n)$, we prove an equivalence between the desingularized values and the renormalized values in Theorem 3.5. As a corollary of Theorem 3.5, we obtain an explicit formula of renormalized values (Corollary 3.9).

3.1. Recurrence formulas among renormalized values. The goal of this subsection is to prove Proposition 3.3 which is on recurrence formula among renormalized values.

We start with the following key lemma of [8] which is a method to compute recursively the image of ϕ_+ (the equation (2.11)).

Lemma 3.1 ([8] Corollary 4.4). *For $w \in Y$ with $\text{dp}(w) > 1$, we have*

$$\phi_+(w) = \frac{1}{2^{\text{dp}(w)} - 2} \sum_{(w)} \phi_+(w') \phi_+(w'').$$

Here we use Sweedler's notation (2.9).

Proposition 3.2. *For $n \in \mathbb{N}_{\geq 2}$ and $k_1, \dots, k_n \in \mathbb{N}_0$, we have*

(3.1)

$$\begin{aligned} \zeta_{\text{EMS}}(-k_1, \dots, -k_n) &= \frac{1}{2^{n-1} - 1} \left\{ \sum_{i_n + j_n = k_n} \binom{k_n}{i_n} \zeta_{\text{EMS}}(-i_n) \zeta_{\text{EMS}}(-k_1, \dots, -k_{n-1} - j_n) \right. \\ &\quad + \sum_{p=2}^{n-1} \sum_{\substack{i_n + j_n = k_n \\ \vdots \\ i_p + j_p = k_p}} \prod_{a=p}^n \binom{k_a}{i_a} \\ &\quad \times \sum_{\substack{\{\circ_q, \diamond_q\} = \{+, \cdot\} \\ p \leq q \leq n-1}} \zeta_{\text{EMS}}(-i_p \circ_p \cdots \circ_{n-1} - i_n) \zeta_{\text{EMS}}(-k_1, \dots, -k_{p-1} - j_p \diamond_p \cdots \diamond_{n-1} - j_n) \Big\}. \end{aligned}$$

Proof. By Proposition 2.5 and Lemma 3.1, for $n \in \mathbb{N}_{\geq 2}$ and $k_1, \dots, k_n \in \mathbb{N}_0$ we get

$$\begin{aligned} \phi_+(w_{k_n} \cdots w_{k_1}) &= \frac{1}{2^{n-1} - 1} \left\{ \sum_{i_n + j_n = k_n} \binom{k_n}{i_n} \phi_+(d^{i_n} y) \phi_+(d^{j_n} w_{k_{n-1}} \cdots w_{k_1}) \right. \\ &\quad + \sum_{p=2}^{n-1} \sum_{\substack{i_n + j_n = k_n \\ \vdots \\ i_p + j_p = k_p}} \prod_{a=p}^n \binom{k_a}{i_a} \sum_{\substack{\{u_q, v_q\} = \{d^{i_q}, d^{j_q} y\} \\ p+1 \leq q \leq n}} \phi_+(u_n \cdots u_{p+1} d^{i_p} y) \phi_+(v_n \cdots v_{p+1} d^{j_p} w_{k_{p-1}} \cdots w_{k_1}) \Big\}, \end{aligned}$$

because $\text{dp}(w) = n$. For $p \leq q \leq n-1$, we define

$$(\circ_q, \diamond_q) := \begin{cases} (+, \flat) & \text{if } (u_{q+1}, v_{q+1}) = (d^{i_{q+1}}, d^{j_{q+1}}y), \\ (\flat, +) & \text{if } (u_{q+1}, v_{q+1}) = (d^{j_{q+1}}y, d^{i_{q+1}}). \end{cases}$$

Then by the definition of $\zeta_{\text{EMS}}(-k_1, \dots, -k_n)$, the equation (3.1) holds. \square

We define the following generating functions in $\mathbb{C}[[x]]$ for $n \in \mathbb{N}_{\geq 2}$ and $k_1, \dots, k_n \in \mathbb{N}_0$:

$$\begin{aligned} \mathfrak{h} &:= \mathfrak{h}(x) := \sum_{k_1=0}^{\infty} \frac{(-x)^{k_1}}{k_1!} \zeta_{\text{EMS}}(-k_1), \\ \mathfrak{h}_{k_1, \dots, k_{n-1}}(x) &:= \sum_{k_n=0}^{\infty} \frac{(-x)^{k_n}}{k_n!} \zeta_{\text{EMS}}(-k_1, \dots, -k_n), \\ \bar{\mathfrak{h}}_{k_1, \dots, k_n}(x) &:= \partial_x^{k_n} \mathfrak{h}_{k_1, \dots, k_{n-1}}(x). \end{aligned}$$

Here for $n \in \mathbb{N}$, we set $\mathfrak{h}_{k_1, \dots, k_{n-1}}(x) := \mathfrak{h}(x)$.

The equation (3.1) looks complicated. But it can be simplified to the following recurrence formula (3.2).

Proposition 3.3. *For $n \in \mathbb{N}_{\geq 2}$ and $k_1, \dots, k_n \in \mathbb{N}_0$, we have*

$$(3.2) \quad \zeta_{\text{EMS}}(-k_1, \dots, -k_n) = \sum_{i_n + j_n = k_n} \binom{k_n}{i_n} \zeta_{\text{EMS}}(-i_n) \zeta_{\text{EMS}}(-k_1, \dots, -k_{n-1} - j_n),$$

and

$$(3.3) \quad \mathfrak{h}_{k_1, \dots, k_{n-1}}(x) = (-1)^{k_1 + \dots + k_{n-1}} (\mathfrak{h} \partial_x^{k_{n-1}}) \dots (\mathfrak{h} \partial_x^{k_1}) (\mathfrak{h}).$$

Proof. We prove (3.2) and (3.3) by induction on $n \in \mathbb{N}_{\geq 2}$. Let $n = 2$. Then by the equation (3.1) of Proposition 3.2, the equation (3.2) clearly holds. And by the equation (3.3) for $n = 2$, we have

$$\begin{aligned} (3.4) \quad \mathfrak{h}_{k_1}(x) &= \sum_{k_2=0}^{\infty} \frac{(-x)^{k_2}}{k_2!} \zeta_{\text{EMS}}(-k_1, -k_2) \\ &= \sum_{k_2=0}^{\infty} \frac{(-x)^{k_2}}{k_2!} \sum_{i_2 + j_2 = k_2} \binom{k_2}{i_2} \zeta_{\text{EMS}}(-i_2) \zeta_{\text{EMS}}(-k_1 - j_2) \\ &= \left\{ \sum_{i_2=0}^{\infty} \frac{(-x)^{i_2}}{i_2!} \zeta_{\text{EMS}}(-i_2) \right\} \left\{ \sum_{j_2=0}^{\infty} \frac{(-x)^{j_2}}{j_2!} \zeta_{\text{EMS}}(-k_1 - j_2) \right\} \\ &= \mathfrak{h} \{ (-1)^{k_1} \partial_x^{k_1} (\mathfrak{h}) \} \\ &= (-1)^{k_1} (\mathfrak{h} \partial_x^{k_1}) (\mathfrak{h}). \end{aligned}$$

Let $n = n_0 \geq 3$. We assume that (3.2) and (3.3) hold for $2 \leq n \leq n_0 - 1$. Firstly, we prove the equation (3.2). By Lemma 3.4 which will be prove later, the second

term of the right hand side of the equation (3.1) is calculated to be

$$\begin{aligned}
& \sum_{p=2}^{n_0-1} \sum_{\{\circ_q, \diamond_q\}=\{+, ,\}} \left\{ \sum_{i_{n_0}+j_{n_0}=k_{n_0}} \binom{k_{n_0}}{i_{n_0}} \zeta_{\text{EMS}}(-i_{n_0}) \zeta_{\text{EMS}}(-k_1, \dots, -k_{n_0-1} - j_{n_0}) \right\} \\
&= \sum_{p=2}^{n_0-1} 2^{n_0-p} \left\{ \sum_{i_{n_0}+j_{n_0}=k_{n_0}} \binom{k_{n_0}}{i_{n_0}} \zeta_{\text{EMS}}(-i_{n_0}) \zeta_{\text{EMS}}(-k_1, \dots, -k_{n_0-1} - j_{n_0}) \right\} \\
&= (2^{n_0-1} - 2) \sum_{i_{n_0}+j_{n_0}=k_{n_0}} \binom{k_{n_0}}{i_{n_0}} \zeta_{\text{EMS}}(-i_{n_0}) \zeta_{\text{EMS}}(-k_1, \dots, -k_{n_0-1} - j_{n_0}).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
& (\text{RHS of (3.1)}) \\
&= \frac{1}{2^{n_0-1} - 1} \left\{ \sum_{i_{n_0}+j_{n_0}=k_{n_0}} \binom{k_{n_0}}{i_{n_0}} \zeta_{\text{EMS}}(-i_{n_0}) \zeta_{\text{EMS}}(-k_1, \dots, -k_{n_0-1} - j_{n_0}) \right. \\
&\quad \left. + (2^{n_0-1} - 2) \sum_{i_{n_0}+j_{n_0}=k_{n_0}} \binom{k_{n_0}}{i_{n_0}} \zeta_{\text{EMS}}(-i_{n_0}) \zeta_{\text{EMS}}(-k_1, \dots, -k_{n_0-1} - j_{n_0}) \right\} \\
&= \sum_{i_{n_0}+j_{n_0}=k_{n_0}} \binom{k_{n_0}}{i_{n_0}} \zeta_{\text{EMS}}(-i_{n_0}) \zeta_{\text{EMS}}(-k_1, \dots, -k_{n_0-1} - j_{n_0}).
\end{aligned}$$

So we get the equation (3.2) for $n \geq 3$.

Secondly, we prove the equation (3.3) for $n = n_0 \geq 3$. By using the equation (3.2) for $n = n_0$ which we have proved just above, we have

$$\mathfrak{h}_{k_1, \dots, k_{n_0-1}}(x) = (-1)^{k_{n_0-1}} \mathfrak{h}(x) \partial_x^{k_{n_0-1}} (\mathfrak{h}_{k_1, \dots, k_{n_0-2}}(x))$$

in the same way to case of $n = 2$. By our induction hypotheses,

$$\begin{aligned}
&= (-1)^{k_{n_0-1}} \mathfrak{h}(x) \partial_x^{k_{n_0-1}} \left((-1)^{k_1 + \dots + k_{n_0-2}} (\mathfrak{h}(x) \partial_x^{k_{n_0-2}}) \dots (\mathfrak{h}(x) \partial_x^{k_1}) (\mathfrak{h}(x)) \right) \\
&= (-1)^{k_1 + \dots + k_{n_0-1}} (\mathfrak{h}(x) \partial_x^{k_{n_0-1}}) \dots (\mathfrak{h}(x) \partial_x^{k_1}) (\mathfrak{h}(x))
\end{aligned}$$

So we get the equation (3.3) for $n \geq 3$. \square

We prove the following lemma used in the above poof.

Lemma 3.4. *Let $n_0 \geq 3$. We assume that (3.3) holds for $n = l$ with $2 \leq l \leq n_0 - 1$. Let $2 \leq p \leq n_0 - 1$ and $\circ_i \in \{+, ,\}$ for $p \leq i \leq n_0 - 1$. Then we have*

$$\begin{aligned}
(3.5) \quad & \sum_{\substack{i_p + j_p = k_p \\ \vdots \\ i_{n_0} + j_{n_0} = k_{n_0}}} \prod_{a=p}^{n_0} \binom{k_a}{i_a} \zeta_{\text{EMS}}(-i_p \circ_p \dots \circ_{n_0-1} - i_{n_0}) \zeta_{\text{EMS}}(-k_1, \dots, -k_{p-1} - j_p \diamond_p \dots \diamond_{n_0-1} - j_{n_0}) \\
&= \sum_{i_{n_0} + j_{n_0} = k_{n_0}} \binom{k_{n_0}}{i_{n_0}} \zeta_{\text{EMS}}(-i_{n_0}) \zeta_{\text{EMS}}(-k_1, \dots, -k_{n_0-1} - j_{n_0}).
\end{aligned}$$

Here \diamond_i is chosen to be with $\{\circ_i, \diamond_i\} = \{+, ,\}$ for $p \leq i \leq n_0 - 1$.

Proof. We get

$$\sum_{k_{n_0}=0}^{\infty} \frac{(-x)^{k_{n_0}}}{k_{n_0}!} (\text{RHS of (3.5)}) = (-1)^{k_{n_0}-1} \mathfrak{h} \partial_x^{k_{n_0}-1} \left(\mathfrak{h}_{k_1, \dots, k_{n_0-2}}(x) \right)$$

in the same way to the computations of $\mathfrak{h}_{k_1}(x)$ in (3.4). By our induction hypothesis on (3.3), for n_0 we obtain

$$(3.6) \quad = (-1)^{k_1 + \dots + k_{n_0-1}} \left(\mathfrak{h} \partial_x^{k_{n_0}-1} \right) \dots \left(\mathfrak{h} \partial_x^{k_1} \right) (\mathfrak{h}).$$

On the other hand, we have

$$\begin{aligned} & \sum_{k_{n_0}=0}^{\infty} \frac{(-x)^{k_{n_0}}}{k_{n_0}!} (\text{LHS of (3.5)}) \\ &= \sum_{\substack{i_p + j_p = k_p \\ \vdots \\ i_{n_0-1} + j_{n_0-1} = k_{n_0-1}}} \prod_{a=p}^{n_0-1} \binom{k_a}{i_a} \left\{ \sum_{i_{n_0}=0}^{\infty} \frac{(-x)^{i_{n_0}}}{i_{n_0}!} \zeta_{\text{EMS}}(-i_p \circ_p \dots \circ_{n_0-1} - i_{n_0}) \right\} \\ & \quad \times \left\{ \sum_{j_{n_0}=0}^{\infty} \frac{(-x)^{j_{n_0}}}{j_{n_0}!} \zeta_{\text{EMS}}(-k_1, \dots, -k_{p-1} - j_p \diamond_p \dots \diamond_{n_0-1} - j_{n_0}) \right\}. \end{aligned}$$

We also consider the following two cases:

Case i) : When $(\circ_{n_0-1}, \diamond_{n_0-1}) = (\circ, +)$, we compute

$$\begin{aligned} & \sum_{k_{n_0}=0}^{\infty} \frac{(-x)^{k_{n_0}}}{k_{n_0}!} (\text{LHS of (3.5)}) \\ &= \sum_{\substack{i_p + j_p = k_p \\ \vdots \\ i_{n_0-1} + j_{n_0-1} = k_{n_0-1}}} \prod_{a=p}^{n_0-1} \binom{k_a}{i_a} \left\{ \sum_{i_{n_0}=0}^{\infty} \frac{(-x)^{i_{n_0}}}{i_{n_0}!} \zeta_{\text{EMS}}(-i_p \circ_p \dots \circ_{n_0-2} - i_{n_0-1}, -i_{n_0}) \right\} \\ & \quad \times \left\{ \sum_{j_{n_0}=0}^{\infty} \frac{(-x)^{j_{n_0}}}{j_{n_0}!} \zeta_{\text{EMS}}(-k_1, \dots, -k_{p-1} - j_p \diamond_p \dots \diamond_{n_0-2} - j_{n_0-1} - j_{n_0}) \right\}. \end{aligned}$$

$$\text{Put } m := \begin{cases} p-1 & \text{when } \diamond_i \text{ is } + \text{ for all } i, \\ \max\{l \mid p \leq l \leq n_0-2, \diamond_l = \circ\} & \text{otherwise.} \end{cases}$$

Then we have

$$\begin{aligned} &= \sum_{\substack{i_p + j_p = k_p \\ \vdots \\ i_{n_0-1} + j_{n_0-1} = k_{n_0-1}}} \prod_{a=p}^{n_0-1} \binom{k_a}{i_a} \left\{ \sum_{i_{n_0}=0}^{\infty} \frac{(-x)^{i_{n_0}}}{i_{n_0}!} \zeta_{\text{EMS}}(-i_p \circ_p \dots \circ_{n_0-2} - i_{n_0-1}, -i_{n_0}) \right\} \\ & \quad \times (-1)^S \partial_x^S \left\{ \sum_{j_{n_0}=0}^{\infty} \frac{(-x)^{j_{n_0}}}{j_{n_0}!} \zeta_{\text{EMS}}(-k_1, \dots, -k_{p-1} - j_p \diamond_p \dots \diamond_{m-1} - j_m) \right\}. \end{aligned}$$

Here $S := \begin{cases} k_{p-1} + j_p + \cdots + j_{n_0-1} & \text{when } \diamond_i \text{ is } + \text{ for all } i, \\ j_{m+1} + \cdots + j_{n_0-1} & \text{otherwise.} \end{cases}$

$$= \sum_{\substack{i_p + j_p = k_p \\ \vdots \\ i_{n_0-1} + j_{n_0-1} = k_{n_0-1}}} (-1)^S \prod_{a=p}^{n_0-1} \binom{k_a}{i_a} \mathfrak{h}_{i_p \circ_p \cdots \circ_{n_0-2} i_{n_0-1}}(x) \cdot \bar{\mathfrak{h}}_{k_1, \dots, k_{p-1} + j_p \diamond_p \cdots \diamond_{n_0-2} j_{n_0-1}}(x).$$

Here we use the definitions of $\mathfrak{h}_{k_1, \dots, k_{n-1}}(x)$ and $\bar{\mathfrak{h}}_{k_1, \dots, k_{n_0}}(x)$. And by using our induction hypothesis on (3.3), we have

$$= \sum_{\substack{i_p + j_p = k_p \\ \vdots \\ i_{n_0-1} + j_{n_0-1} = k_{n_0-1}}} \prod_{a=p}^{n_0-1} \binom{k_a}{i_a} \left\{ (-1)^{\sum_{q=p}^{n_0-1} i_q} \left(\mathfrak{h} \partial_x^{i_{n_0-1}} \right) \left(\mathfrak{h}^{\delta_{n_0-2}} \partial_x^{i_{n_0-2}} \right) \cdots \left(\mathfrak{h}^{\delta_p} \partial_x^{i_p} \right) (\mathfrak{h}) \right\} \\ \times \left\{ (-1)^{\sum_{q=1}^{p-1} k_q + \sum_{q=p}^{n_0-1} j_q} \partial_x^{j_{n_0-1}} \left(\mathfrak{h}^{1-\delta_{n_0-2}} \partial_x^{j_{n_0-2}} \right) \cdots \left(\mathfrak{h}^{1-\delta_p} \partial_x^{j_p} \right) (\mathfrak{h} \partial_x^{k_{p-1}}) \cdots (\mathfrak{h} \partial_x^{k_1}) (\mathfrak{h}) \right\}.$$

Here we put $\delta_i := \begin{cases} 0 & \text{if } \circ_i = +, \\ 1 & \text{if } \circ_i = , \end{cases}$ for $p \leq i \leq n_0 - 2$.

$$= (-1)^{\sum_{q=1}^{n_0-1} k_q} \mathfrak{h} \sum_{\substack{i_p + j_p = k_p \\ \vdots \\ i_{n_0-1} + j_{n_0-1} = k_{n_0-1}}} \prod_{a=p}^{n_0-1} \binom{k_a}{i_a} \left\{ \partial_x^{i_{n_0-1}} \left(\mathfrak{h}^{\delta_{n_0-2}} \partial_x^{i_{n_0-2}} \right) \cdots \left(\mathfrak{h}^{\delta_p} \partial_x^{i_p} \right) (\mathfrak{h}) \right\} \\ \times \left\{ \partial_x^{j_{n_0-1}} \left(\mathfrak{h}^{1-\delta_{n_0-2}} \partial_x^{j_{n_0-2}} \right) \cdots \left(\mathfrak{h}^{1-\delta_p} \partial_x^{j_p} \right) (\mathfrak{h} \partial_x^{k_{p-1}}) \cdots (\mathfrak{h} \partial_x^{k_1}) (\mathfrak{h}) \right\} \\ = (-1)^{\sum_{q=1}^{n_0-1} k_q} \mathfrak{h} \partial_x^{k_{n_0-1}} \left(\mathfrak{h} \sum_{\substack{i_p + j_p = k_p \\ \vdots \\ i_{n_0-2} + j_{n_0-2} = k_{n_0-2}}} \prod_{a=p}^{n_0-2} \binom{k_a}{i_a} \left\{ \partial_x^{i_{n_0-2}} \left(\mathfrak{h}^{\delta_{n_0-3}} \partial_x^{i_{n_0-3}} \right) \cdots \left(\mathfrak{h}^{\delta_p} \partial_x^{i_p} \right) (\mathfrak{h}) \right\} \right. \\ \left. \times \left\{ \partial_x^{j_{n_0-2}} \left(\mathfrak{h}^{1-\delta_{n_0-3}} \partial_x^{j_{n_0-3}} \right) \cdots \left(\mathfrak{h}^{1-\delta_p} \partial_x^{j_p} \right) (\mathfrak{h} \partial_x^{k_{p-1}}) \cdots (\mathfrak{h} \partial_x^{k_1}) (\mathfrak{h}) \right\} \right).$$

We use Leibniz rule in last equality. By using this rule repeatedly, we get

$$= (-1)^{\sum_{q=1}^{n_0-1} k_q} \left(\mathfrak{h} \partial_x^{k_{n_0-1}} \right) \cdots (\mathfrak{h} \partial_x^{k_1}) (\mathfrak{h}).$$

This is equal to (3.6).

Case ii) : When $(\circ_{n_0-1}, \diamond_{n_0-1}) = (+, ,)$, it can be proved in the same way to *Case i)*.

□

3.2. An equivalence between desingularized values and renormalized ones.

We reveal a close relationship among desingularized values and renormalized ones in Theorem 3.5. As a consequence, we get an explicit formula of renormalized values in terms of Bernoulli numbers in Corollary 3.9.

Our main theorem of this paper is the following explicit relationship between the generating function $Z_{\text{FKMT}}(t_1, \dots, t_n)$ of the desingularized values $\zeta_{\text{FKMT}}(-k_1, \dots, -k_n)$ in (0.4) and the generating function $Z_{\text{EMS}}(t_1, \dots, t_n)$ of the renormalized values $\zeta_{\text{EMS}}(-k_1, \dots, -k_n)$ in (0.6).

Theorem 3.5. *For $n \in \mathbb{N}$, we have*

$$(3.7) \quad Z_{\text{EMS}}(t_1, \dots, t_n) = \prod_{i=1}^n \frac{1 - e^{-t_i - \dots - t_n}}{t_i + \dots + t_n} \cdot Z_{\text{FKMT}}(-t_1, \dots, -t_n).$$

Proof. By Proposition 3.3 and Lemma 3.4 we get

$$(3.8) \quad \zeta_{\text{EMS}}(-k_1, \dots, -k_n) = \sum_{\substack{i_2 + j_2 = k_2 \\ \vdots \\ i_n + j_n = k_n}} \prod_{a=2}^n \binom{k_a}{i_a} \zeta_{\text{EMS}}(-i_2, \dots, -i_n) \zeta_{\text{EMS}}(-k_1 - j_2 - \dots - j_n).$$

Here, we use Lemma 3.4 for $p = 2$ and for all $\circ_q = \bullet$ ($2 \leq q \leq n$). It is remarkable that the same recurrence formula holds for $\zeta_{\text{FKMT}}(-k_1, \dots, -k_n)$ of (1.5). Thus, we get

$$(3.9) \quad Z_{\text{EMS}}(t_1, \dots, t_n) = Z_{\text{EMS}}(t_2, \dots, t_n) \cdot Z_{\text{EMS}}(t_1 + \dots + t_n) \quad (n \in \mathbb{N}).$$

Now from [8] Theorem 4.3, $\zeta_{\text{EMS}}(-k_1) = \zeta(-k_1)$ at $k_1 \in \mathbb{N}_0$, so we can write $Z_{\text{EMS}}(x)$ by

$$Z_{\text{EMS}}(x) = \frac{1 + x - e^x}{x(e^x - 1)}.$$

We get the following equation by $Z_{\text{EMS}}(x)$ and $Z_{\text{FKMT}}(x)$:

$$(3.10) \quad Z_{\text{EMS}}(x) = \frac{1 - e^{-x}}{x} Z_{\text{FKMT}}(-x).$$

By using (1.4), (3.9) and (3.10), we get (3.7). \square

By Theorem 3.5, we find that desingularized values and renormalized ones are equivalent. Namely, the renormalized values can be given as linear combinations of the desingularized ones.

Examples 3.6. The desingularized values and the renormalized values are equal at the origin:

$$\zeta_{\text{FKMT}}(\underbrace{0, \dots, 0}_n) = \zeta_{\text{EMS}}(\underbrace{0, \dots, 0}_n) = B_1^n = \left(-\frac{1}{2}\right)^n$$

Examples 3.7. For $k_1, k_2, k_3 \in \mathbb{N}_0$, we have

$$\begin{aligned}\zeta_{\text{EMS}}(-k_1) &= \sum_{\nu_{01}+\nu_{11}=k_1} \binom{k_1}{\nu_{01}} \frac{(-1)^{\nu_{11}}}{\nu_{01}+1} \zeta_{\text{FKMT}}(-\nu_{11}), \\ \zeta_{\text{EMS}}(-k_1, -k_2) &= \sum_{\substack{\nu_{01}+\nu_{11}=k_1 \\ \nu_{02}+\nu_{12}+\nu_{22}=k_2}} \binom{k_1}{\nu_{01}} \binom{k_2}{\nu_{02} \ \nu_{12}} \frac{1}{\nu_{02}+1} \frac{(-1)^{\nu_{11}+\nu_{22}}}{\nu_{01}+\nu_{12}+1} \zeta_{\text{FKMT}}(-\nu_{11}, -\nu_{22}), \\ \zeta_{\text{EMS}}(-k_1, -k_2, -k_3) &= \sum_{\substack{\nu_{01}+\nu_{11}=k_1 \\ \nu_{02}+\nu_{12}+\nu_{22}=k_2 \\ \nu_{03}+\nu_{13}+\nu_{23}+\nu_{33}=k_3}} \binom{k_1}{\nu_{01}} \binom{k_2}{\nu_{02} \ \nu_{12}} \binom{k_3}{\nu_{03} \ \nu_{13} \ \nu_{23}} \\ &\quad \times \frac{1}{\nu_{03}+1} \frac{1}{\nu_{02}+\nu_{13}+1} \frac{(-1)^{\nu_{01}+\nu_{12}+\nu_{23}}}{\nu_{01}+\nu_{12}+\nu_{23}+1} \zeta_{\text{FKMT}}(-\nu_{11}, -\nu_{22}, -\nu_{33}).\end{aligned}$$

Here $\binom{k_2}{\nu_{02} \ \nu_{12}} := \frac{k_2!}{\nu_{02}! \nu_{12}! (k_2 - \nu_{02} - \nu_{12})!}$ and $\binom{k_3}{\nu_{03} \ \nu_{13} \ \nu_{23}} := \frac{k_3!}{\nu_{03}! \nu_{13}! \nu_{23}! (k_3 - \nu_{03} - \nu_{13} - \nu_{23})!}$.

On the other hand, desingularized values can be also given as linear combinations of product of renormalized ones and Bernoulli numbers B_n :

Examples 3.8. For $k_1, k_2, k_3 \in \mathbb{N}_0$, we have

$$\begin{aligned}\zeta_{\text{FKMT}}(-k_1) &= (-1)^{k_1} \sum_{\nu_{01}+\nu_{11}=k_1} \binom{k_1}{\nu_{01}} B_{\nu_{01}} \zeta_{\text{EMS}}(-\nu_{11}), \\ \zeta_{\text{FKMT}}(-k_1, -k_2) &= (-1)^{k_1+k_2} \sum_{\substack{\nu_{01}+\nu_{11}=k_1 \\ \nu_{02}+\nu_{12}+\nu_{22}=k_2}} \binom{k_1}{\nu_{01}} \binom{k_2}{\nu_{02} \ \nu_{12}} B_{\nu_{02}} B_{\nu_{01}+\nu_{12}} \zeta_{\text{EMS}}(-\nu_{11}, -\nu_{22}), \\ \zeta_{\text{FKMT}}(-k_1, -k_2, -k_3) &= (-1)^{k_1+k_2+k_3} \sum_{\substack{\nu_{01}+\nu_{11}=k_1 \\ \nu_{02}+\nu_{12}+\nu_{22}=k_2 \\ \nu_{03}+\nu_{13}+\nu_{23}+\nu_{33}=k_3}} \binom{k_1}{\nu_{01}} \binom{k_2}{\nu_{02} \ \nu_{12}} \binom{k_3}{\nu_{03} \ \nu_{13} \ \nu_{23}} \\ &\quad \times B_{\nu_{03}} B_{\nu_{02}+\nu_{13}} B_{\nu_{01}+\nu_{12}+\nu_{23}} \zeta_{\text{EMS}}(-\nu_{11}, -\nu_{22}, -\nu_{33}).\end{aligned}$$

By combining Proposition 1.5 and Theorem 3.5, we obtain the following corollary.

Corollary 3.9. For $n \in \mathbb{N}$, we have

$$Z_{\text{EMS}}(t_1, \dots, t_n) = \prod_{i=1}^n \frac{(t_i + \dots + t_n) + (1 - e^{t_i + \dots + t_n})}{(t_i + \dots + t_n)(1 - e^{t_i + \dots + t_n})}.$$

The above equation is equivalent to the equation (0.7). Therefore the renormalized values are described explicitly in terms of Bernoulli numbers:

Examples 3.10. For $k_1, k_2, k_3 \in \mathbb{N}_0$, we have

$$\begin{aligned}\zeta_{\text{EMS}}(-k_1) &= \frac{(-1)^{k_1}}{k_1 + 1} B_{k_1+1}, \\ \zeta_{\text{EMS}}(-k_1, -k_2) &= (-1)^{k_1+k_2} \sum_{\nu_{12}+\nu_{22}=k_2} \binom{k_2}{\nu_{12}} \frac{B_{\nu_{22}+1}}{\nu_{22}+1} \frac{B_{k_1+\nu_{12}+1}}{k_1+\nu_{12}+1}, \\ \zeta_{\text{EMS}}(-k_1, -k_2, -k_3) &= (-1)^{k_1+k_2+k_3} \sum_{\substack{\nu_{12}+\nu_{22}=k_2 \\ \nu_{13}+\nu_{23}+\nu_{33}=k_3}} \binom{k_2}{\nu_{12}} \binom{k_3}{\nu_{13} \ \nu_{23}} \\ &\quad \times \frac{B_{\nu_{33}+1}}{\nu_{33}+1} \frac{B_{\nu_{22}+\nu_{23}+1}}{\nu_{22}+\nu_{23}+1} \frac{B_{k_1+\nu_{12}+\nu_{13}+1}}{k_1+\nu_{12}+\nu_{13}+1}.\end{aligned}$$

As is explained in our introduction, other types of renormalized values were investigated in several literatures ([9], [13], [17] etc). However, their explicit relationships with the desingularized values $\zeta_{\text{FKMT}}(-k_1, \dots, -k_n)$ do not seem to be shown so far, actually which was posed as a question in [11] Question 4.8. It would be great if our equivalence (Theorem 3.5) could also lead a direction to settle their question.

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